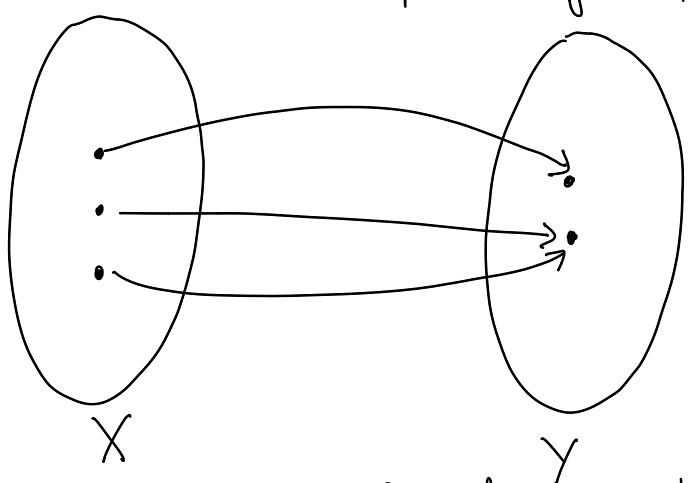
§ 2. Functions

We know functions as maps from the set of real numbers $\mathbb{R}$ to $\mathbb{R}$, e.g.

$$y = f(x) = x - x^3, \quad -1 \le x \le 1$$

More generally, we have for sets X, Y:

Definition 2.1:
A "Function" $f: X \longrightarrow Y$ identifies with every point $x \in X$ exactly one "image" $y = f(x) \in Y$. Every $z \in X$ with $y = f(z)$ is then denoted as "preimage" of y.



Thus a function is defined through the specification of:

- a "domain" (here X)
- an "image" (here Y)
- a "mapping rule" (here $x \longmapsto f(x)$)

Example 2.1:

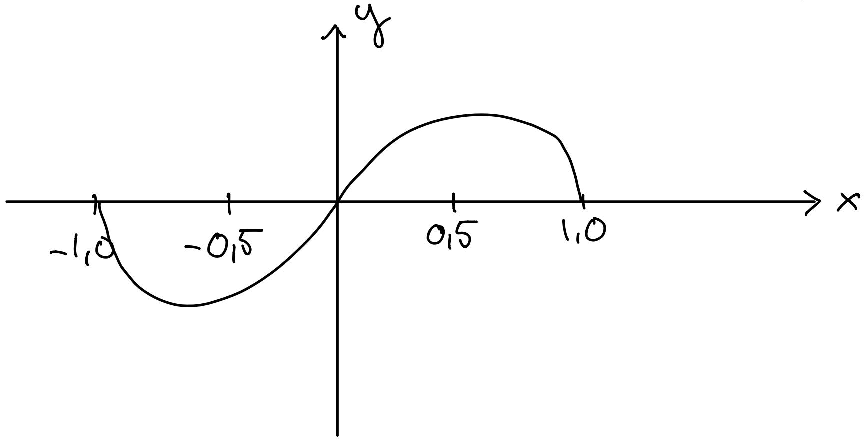
i) $f: \mathbb{R} \longrightarrow \mathbb{R}$; $x \longmapsto x - x^3$

ii) $f: [-1,1] \longrightarrow \mathbb{R}$; $x \longmapsto x - x^3$;

iv) $h: \mathbb{R} \longrightarrow [0, \infty)$, $x \longmapsto x^2$;

v) $id_X: X \longrightarrow X$, $x \longmapsto x = id_X(x)$;
the identity map on X.

We can represent functions through their "graph":



§2.1 Composition of maps

Let $f: X \longrightarrow Y$, $g: Y \longrightarrow Z$ be maps
Through composition we obtain a new map

$$g \circ f: X \longrightarrow Z, \quad x \longmapsto g(f(x))$$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "F = g \circ f"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

Proposition 2.1:

For maps $f: X \longrightarrow Y$, $g: Y \longrightarrow Z$, $h: Z \longrightarrow W$ we have

$$h \circ (g \circ f) = (h \circ g) \circ f \quad \text{(associative law)}$$

Proof:

i) The domains X are identical

ii) The images Y are identical

iii) We show equality of the mapping rule:

$\forall x \in X$ we have

$$(h \circ (g \circ f))(x) = h((g \circ f)(x)) = h(g(f(x)))$$
$$= h \circ g(f(x)) = ((h \circ g) \circ f)(x)$$

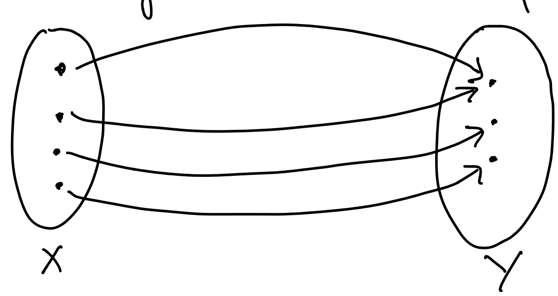
$\square$

Definition 2.2:

Let $f: X \longrightarrow Y$ be a map.

i) f is "surjective" (or "onto"), if for each $y \in Y$ there is at least one preimage, i.e.

$$\forall y \in Y \; \exists x \in X : f(x) = y.$$

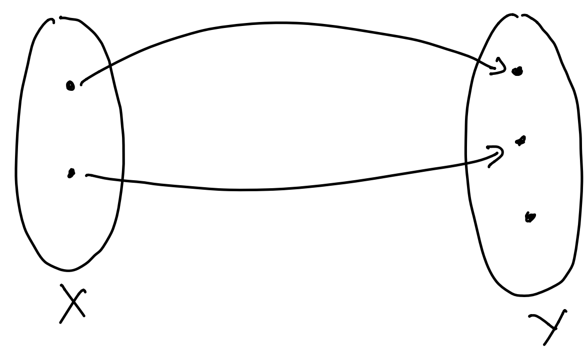


f surjective

ii) f is "injective" (or "one-to-one"), if for each $y \in Y$ there is <u>at most</u> one preimage, i.e

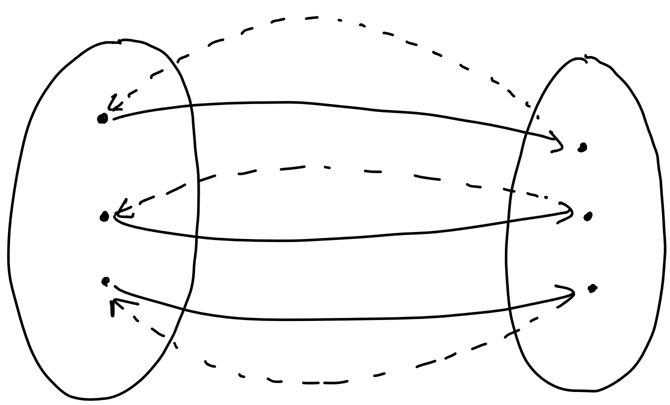
$$\forall \; x_1, x_2 \in X: \; f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

$$\Leftrightarrow \; \forall \; x_1, x_2 \in X: \; x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$$



f injective

iii) f is "bijective", if for each $y \in Y$ there is <u>exactly</u> one preimage, i.e. f is onto <u>and</u> one-to-one.



f is bijective with inverse g

If f is bijective, then there exists an "inverse map" $g: Y \to X$ with the property

$$g \circ f = id_X, \quad f \circ g = id_Y$$

We also write $g = f^{-1}$.

Example 2.2:

i) The map $f: \mathbb{R} \longrightarrow \mathbb{R};\ x \longmapsto x - x^3$ is surjective but not injective

ii) The map $f: (-\pi/2, \pi/2) \longrightarrow (-1, 1)$ with $x \longmapsto \sin(x)$ is bijective

Proposition 2.2:

Let $f: X \longrightarrow Y$ be a map. Then

i) f is injective $\Rightarrow \exists\ g: Y \longrightarrow X$ with $g \circ f = id_X$

ii) f is surjective $\Rightarrow \exists\ g: Y \longrightarrow X$ with $f \circ g = id_Y$

iii) f is bijective $\Rightarrow \exists\ g: Y \rightarrow X$ with $f \circ g = id_Y$, $g \circ f = id_X$

Proof:

i) $\forall\ y \in f(X) = \{f(x) \mid x \in X\} \subset Y$ there is exactly one preimage $x := g(y)$

Set $g(y) = x_0 \in X$ for $y \notin f(X)$

Then $g: Y \rightarrow X$ is well-defined and $g \circ f = id_X$

ii) For $y \in Y$ we have $A(y) = \{x \in X \mid f(x) = y\} \neq \emptyset$.

Choose arbitrary $x \in A(y)$ and set $g(y) = x$

$\Rightarrow g: Y \rightarrow X$ is well-defined and $f \circ g = id_Y$. $\square$

§3. Sequences and series

§3.1 Examples

i) "Fibonacci numbers"

$1, 1, 2, 3, 5, 8, 13, \ldots$

arise from a simple population model from the law

$$a_0 = 1, \quad a_1 = 1, \quad a_{n+1} = a_n + a_{n-1}, \quad \forall n \in \mathbb{N}$$

ii) The numbers (from interest rates)

$$a_n = \left(1 + \frac{1}{n}\right)^n, \quad n \in \mathbb{N}$$

approach for $n \longrightarrow \infty$ the euler number

$$e = 2.718 \cdots \qquad \text{(Definition of euler number)}$$

iii) The geometric series

$$S_n = 1 + q + q^2 + \cdots + q^n = \sum_{k=0}^{n} q^k, \quad n \in \mathbb{N},$$

has for $-1 < q < 1$ the "limit" $S = \frac{1}{1-q}$

§3.2 Limit of a sequence

In fifth century BC the Greek philosopher Zeno asked: Who will win in a race between Achilles and a tortoise ?

The paradox can be explained by noting that the sequences $(a_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ (positions of Achilles and positions of the tortoise) approach a common "limit", namely the point were Achilles overtakes the tortoise.

$\longrightarrow$ Let $(a_n)_{n\in\mathbb{N}} = (a_1, a_2, a_3, \dots)$ be a sequence in $\mathbb{R}$, $a \in \mathbb{R}$.

Definition 3.1:

i) The sequence $(a_n)_{n\in\mathbb{N}}$ "converges" to a for $n \longrightarrow \infty$, if

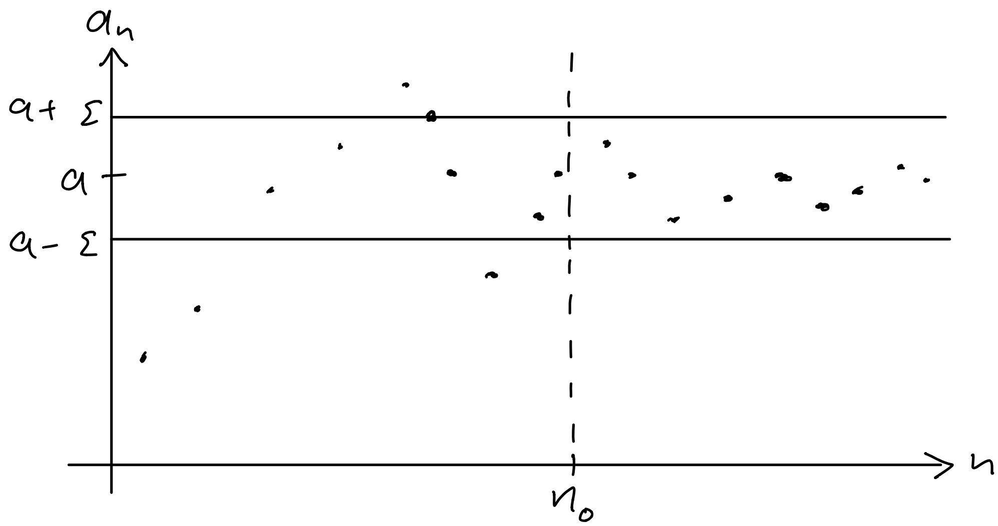
$$\forall \varepsilon > 0 \ \exists \ n_0 = n_0(\varepsilon) \in \mathbb{N} \ \forall n \geq n_0 : |a_n - a| < \varepsilon;$$

We then write

$$a = \lim_{n\to\infty} a_n \quad \text{or} \quad a_n \longrightarrow a \ (n \longrightarrow \infty)$$

and call a the "limit" of the sequence $(a_n)_{n\in\mathbb{N}}$.

ii) A sequence $(a_n)_{n\in\mathbb{N}}$ is called "convergent", if it has a limit; otherwise the sequence is called "divergent".



Example 3.1:

i) For $a_n = \frac{1}{n}$, $n \in \mathbb{N}$, $a_n \longrightarrow 0 \ (n \to \infty)$

ii) Let $q \in \mathbb{R}$ with $0 < q < 1$. Then $q^n \to 0$ $(n \to \infty)$

Lemma 3.1 (Bernoulli inequality)

Let $x > -1$. Then we have:

$$\forall \ n \in \mathbb{N}: (1+x)^n \geq 1 + nx$$

Proof:

Induction:

a) $n = 1$ ✓

b) $n \mapsto n+1$:

assume $(1+x)^n \geq 1 + nx$ ("induction assumption")

$$\Longrightarrow (1+x)^{n+1} = (1+x)^n \underbrace{(1+x)}_{\geq 0} \geq (1+nx)(1+x)$$

$$= 1 + (n+1)x + \underbrace{n^2 x}_{\geq 0} \geq 1 + (n+1)x$$

$\square$

<u>Proof of Example 3.1:</u>

i) For each $\varepsilon > 0$ there is an $n_0 \in \mathbb{N}$ with $n_0 > \frac{1}{\varepsilon}$ or $\frac{1}{n_0} < \varepsilon$

$$\Rightarrow \forall\ n \geq n_0 : -\varepsilon < 0 < \frac{1}{n} \leq \frac{1}{n_0} < \varepsilon$$

ii) Write $\frac{1}{q} = 1 + \delta$ with $\delta > 0$. The Bernoulli inequality then gives

$$\forall\ n \in \mathbb{N} : \frac{1}{q^n} = \left(\frac{1}{q}\right)^n = (1+\delta)^n \geq 1 + n\delta \geq n\delta$$

$$\Rightarrow \forall\ n \in \mathbb{N} : \quad 0 < q^n \leq \frac{1}{n\delta}$$

For $\varepsilon > 0$ choose $n_0 = n_0(\varepsilon)$ with $\frac{1}{n_0} < \varepsilon\delta$

Then $\forall\ n \geq n_0 : \ 0 < q^n \leq \frac{1}{n\delta} \leq \frac{1}{n_0 \delta} < \varepsilon$

$\square$

<u>Proposition 3.1</u> (triangle-inequality)

$$\forall\ x, y \in \mathbb{R} : \ |x+y| \leq |x| + |y|$$

<u>Proof:</u>

For every $x \in \mathbb{R}$ we have $x \leq |x|$, $-x \leq |x|$.

Therefore, $\quad x + y \leq |x| + |y|, \ -(x+y) = (-x) + (-y) \leq |x| + |y|$

$\square$

Example 3.2:

Not every sequence $(a_n)_{n\in\mathbb{N}}$ converges, as the following examples show:

i) Let $a_n = (-1)^n$, $n \in \mathbb{N}$. Then for each $a \in \mathbb{R}$, $n \in \mathbb{N}$ we have from Prop. 3.1:

$$|a_n - a| + |a_{n+1} - a| \geq |(a_n - a) - (a_{n+1} - a)| = 2,$$

and no $a \in \mathbb{R}$ can be the limit of $(a_n)_{n\in\mathbb{N}}$

ii) Let $a_n = n$, $n \in \mathbb{N}$. For each $a \in \mathbb{R}$ there is an $n_0 \in \mathbb{N}$ such that $a < n_0$

$\Rightarrow \forall \; n \geq n_0 : |a_n - a| = n - a \geq n_0 - a > 0$

$\Rightarrow$ a is not limit of (a_n).

iii) Fibonacci-numbers:

$F_0 = 1, \; F_1 = 1, \; F_{n+1} = F_n + F_{n-1} \quad (n \geq 2)$

induction $\Rightarrow F_n \geq n, \; n \in \mathbb{N}$

$\Rightarrow$ no limit.

$\square$

Proposition 3.2:

Let $(a_n)_{n\in\mathbb{N}}$ converge to $a \in \mathbb{R}$ as well as to $b \in \mathbb{R}$.

Then $a = b$.

Proof:

Assume $a \neq b$. Choose $\varepsilon = \frac{|b-a|}{2} > 0$ and $n_0 = n_0(\varepsilon)$ with

$$\forall\ n \geq n_0: \ |a_n - a| < \varepsilon, \ |a_n - b| < \varepsilon$$

Then we have for $n \geq n_0$:

$$2\varepsilon = |a-b| = |(a-a_n)-(b-a_n)|$$
$$\leq |a-a_n| + |b-a_n| < 2\varepsilon$$

$\lightning$ contradiction

$\square$

Proposition 3.3 (computation with limits)

Let the sequences $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ be convergent with $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$.

Then the sequences $(a_n + b_n)_{n\in\mathbb{N}}$, $(a_n \cdot b_n)_{n\in\mathbb{N}}$ converge and

i) $\lim_{n\to\infty}(a_n + b_n) = a + b = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$,

ii) $\lim_{n\to\infty}(a_n \cdot b_n) = a \cdot b = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$,

iii) If in addition $b \neq 0 \neq b_n$ for all n, then we have $\lim_{n\to\infty}(a_n/b_n) = a/b$

iv) If $a_n \leq b_n$ for $n \in \mathbb{N}$, then also $a \leq b$

Remark 3.1:

If $a_n < b_n$, $n \in \mathbb{N}$, then it does in general not follow that $a < b$.

Example

$$a_n := 0 < \frac{1}{n} =: b_n, \quad n \in \mathbb{N},$$

$$\text{with } \lim_{n\to\infty} a_n = 0 = \lim_{n\to\infty} b_n$$

Proof of Prop. 3.3:

For $\varepsilon > 0$ let $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that

$$\forall\ n \geq n_0: \ |a_n - a| < \varepsilon, \ \ |b_n - b| < \varepsilon$$

i) $\forall\ n \geq n_0: \ |(a_n + b_n) - (a+b)| \leq |a_n - a| + |b_n - b| < 2\varepsilon$

ii) Let $\varepsilon < 1$. Then we have

$$\forall\ n \geq n_0: \ |b_n| \leq |b_n - b| + |b| \leq |b| + 1$$

and

$$\forall\ n \geq n_0: \ |a_n b_n - ab| = |(a_n - a) b_n + a(b_n - b)|$$

$$\leq |b_n| \cdot |a_n - a| + |a| \cdot |b_n - b|$$

$$\leq (|a| + |b| + 1)\varepsilon$$

alalogously iii) and iv)

$\square$